

ON THE GEOMETRY OF TENSOR NETWORK STATES

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ABSTRACT. We answer a question of L. Grasedyck that arose in quantum information theory, showing that the limit of tensors in a space of tensor network states need not be a tensor network state. We also give geometric descriptions of spaces of tensor networks states corresponding to trees and loops. Grasedyck's question has a surprising connection to the area of Geometric Complexity Theory, in that the result is equivalent to the statement that the boundary of the Mulmuley-Sohoni type variety associated to matrix multiplication is strictly larger than the projections of matrix multiplication (and re-expressions of matrix multiplication and its projections after changes of bases). Tensor Network States are also related to graphical models in algebraic statistics.

1. INTRODUCTION

1.1. Origin in physics. Tensors describe states of quantum mechanical systems. If a system has n particles, its state is an element of $H_1 \otimes \cdots \otimes H_n$ with H_j Hilbert spaces. In numerical many-body physics, in particular solid state physics, one wants to simulate quantum states of thousands of particles, often arranged on a regular lattice (e.g., atoms in a crystal). Due to the exponential growth of the dimension of $H_1 \otimes \cdots \otimes H_n$ with n , any naïve method of representing these tensors is intractable on a computer. Tensor network states were defined to reduce the complexity of the spaces involved by restricting to a subset of tensors that is physically reasonable, in the sense that the corresponding spaces of tensors are only locally entangled because interactions (entanglement) in the physical world appear to just happen locally.

Such spaces have been studied since the 1980's. These spaces are associated to graphs, and go under different names: *tensor network states*, *finitely correlated states (FCS)*, *valence-bond solids (VBS)*, *matrix product states (MPS)*, *projected entangled pairs states (PEPS)*, and *multi-scale entanglement renormalization ansatz states (MERA)*, see, e.g., [14, 7, 9, 6, 15, 5] and the references therein. We will use the term tensor network states.

1.2. Definitions and notation. For a graph Γ with edges e_s and vertices v_j , $s \in e(j)$ means e_s is incident to v_j . If Γ is directed, $s \in in(j)$ are the incoming edges and $s \in out(j)$ the outgoing edges.

Let V_1, \dots, V_n be complex vector spaces, let $\mathbf{v}_i = \dim V_i$. Let Γ be a graph with n vertices v_j , $1 \leq j \leq n$, and m edges e_s , $1 \leq s \leq m$, and let $\vec{\mathbf{e}} = (\mathbf{e}_1, \dots, \mathbf{e}_m) \in \mathbb{N}^m$. Associate V_j to the vertex v_j and an auxiliary vector space E_s of dimension \mathbf{e}_s to the edge e_s . Make Γ into a directed graph. (The choice of directions will not effect the end result.) Let $\mathbf{V} = V_1 \otimes \cdots \otimes V_n$.

Let

$$(1) \quad TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V}) := \{T \in \mathbf{V} \mid \exists T_j \in V_j \otimes (\otimes_{s \in in(j)} E_s) \otimes (\otimes_{t \in out(j)} E_t^*), \text{ such that } T = Con(T_1 \otimes \cdots \otimes T_n)\}$$

where Con is the contraction of all the E_s 's with all the E_s^* 's.

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Example 1.2.1. Let Γ be a graph with two vertices and one edge connecting them, then, $TNS(\Gamma, \mathbf{e}_1, V_1 \otimes V_2)$ is just the set of elements of $V_1 \otimes V_2$ of rank at most \mathbf{e}_1 , denoted $\hat{\sigma}_{\mathbf{e}_1}(\text{Seg}(\mathbb{P}V_1 \times \mathbb{P}V_2))$ and called the (cone over the) \mathbf{e}_1 -st secant variety of the Segre variety. To see this, let $\epsilon_1, \dots, \epsilon_{\mathbf{e}_1}$ be a basis of E_1 and $\epsilon^1, \dots, \epsilon^{\mathbf{e}_1}$ the dual basis of E^* . Assume, to avoid trivialities, that $\mathbf{v}_1, \mathbf{v}_2 \geq \mathbf{e}_1$. Given $T_1 \in V_1 \otimes E_1$ we may write $T_1 = u_1 \otimes \epsilon_1 + \dots + u_{\mathbf{e}_1} \otimes \epsilon_{\mathbf{e}_1}$ for some $u_\alpha \in V_1$. Similarly, given $T_2 \in V_2 \otimes E_1^*$ we may write $T_2 = w_1 \otimes \epsilon^1 + \dots + w_{\mathbf{e}_1} \otimes \epsilon^{\mathbf{e}_1}$ for some $w_\alpha \in V_2$. Then $\text{Con}(T_1 \otimes T_2) = u_1 \otimes w_1 + \dots + u_{\mathbf{e}_1} \otimes w_{\mathbf{e}_1}$.

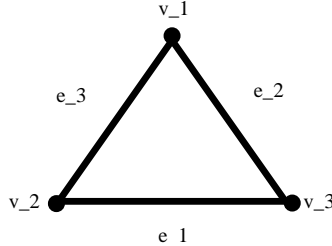
The graph used to define a set of tensor network states is often modeled to mimic the physical arrangement of the particles, with edges connecting nearby particles, as nearby particles are the ones likely to be entangled.

Remark 1.2.2. The construction of tensor network states in the physics literature does not use a directed graph, because all vector spaces are Hilbert spaces, and thus self-dual. However the sets of tensors themselves do not depend on the Hilbert space structure of the vector space, which is why we omit this structure. The small price to pay is the edges of the graph must be oriented, but all orientations lead to the same set of tensor network states.

1.3. Grasedyck's question. Lars Grasedyck asked:

Is $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ Zariski closed? That is, given a sequence of tensors $T_\epsilon \in \mathbf{V}$ that converges to a tensor T_0 , if $T_\epsilon \in TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ for all $\epsilon \neq 0$, can we conclude $T_0 \in TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$?

He mentioned that he could show this to be true when Γ was a tree, but did not know the answer when Γ is a triangle.



Definition 1.3.1. A dimension \mathbf{v}_j is *critical*, resp. *subcritical*, resp. *supercritical*, if $\mathbf{v}_j = \Pi_{s \in e(j)} \mathbf{e}_s$, resp. $\mathbf{v}_j \leq \Pi_{s \in e(j)} \mathbf{e}_s$, resp. $\mathbf{v}_j \geq \Pi_{s \in e(j)} \mathbf{e}_s$. If $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ is critical for all j , we say $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ is critical, and similarly for sub- and super-critical.

Theorem 1.3.2. $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ is not Zariski closed for any Γ containing a cycle whose vertices have non-subcritical dimensions.

Notation. $GL(V)$ denotes the group of invertible linear maps $V \rightarrow V$. $GL(V_1) \times \dots \times GL(V_n)$ acts on $V_1 \otimes \dots \otimes V_n$ by $(g_1, \dots, g_n) \cdot v_1 \otimes \dots \otimes v_n = (g_1 v_1) \otimes \dots \otimes (g_n v_n)$. (Here $v_j \in V_j$ and the action on a tensor that is a sum of rank one tensors is the sum of the actions on the rank one tensors.) Let $\text{End}(V)$ denote the set of all linear maps $V \rightarrow V$. We adopt the convention that $\text{End}(V_1) \times \dots \times \text{End}(V_n)$ acts on $V_1 \otimes \dots \otimes V_n$ by $(Z_1, \dots, Z_n) \cdot v_1 \otimes \dots \otimes v_n = (Z_1 v_1) \otimes \dots \otimes (Z_n v_n)$. Let $\mathfrak{gl}(V)$ denote the Lie algebra of $GL(V)$. It is naturally isomorphic to $\text{End}(V)$ but it acts on $V_1 \otimes \dots \otimes V_n$ via the Leibnitz rule: $(X_1, \dots, X_n) \cdot v_1 \otimes \dots \otimes v_n = (X_1 v_1) \otimes v_2 \otimes \dots \otimes v_n + v_1 \otimes (X_2 v_2) \otimes v_3 \otimes \dots \otimes v_n + \dots + v_1 \otimes \dots \otimes v_{n-1} \otimes (X_n v_n)$. (This is because elements of the Lie algebra should be thought of as derivatives of curves in the Lie group at the identity.) If $X \subset V$ is a subset, $\overline{X} \subset V$ denotes its closure. This closure is the same whether one uses the Zariski closure, which is the common zero set of all polynomials vanishing on X , or

the Euclidean closure, where one fixes a metric compatible with the linear structure on V and takes the closure with respect to limits.

1.4. Connections to the GCT program. The triangle case is especially interesting because we remark below that in the critical dimension case it corresponds to

$$\text{End}(V_1) \times \text{End}(V_2) \times \text{End}(V_3) \cdot \text{Mmult}_{\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1},$$

where, setting $V_1 = E_2^* \otimes E_3$, $V_2 = E_3^* \otimes E_1$, and $V_3 = E_2 \otimes E_1^*$, $\text{Mmult}_{\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1} \in V_1 \otimes V_2 \otimes V_3$ is the matrix multiplication operator, that is, as a tensor, $\text{Mmult}_{\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1} = \text{Id}_{E_3} \otimes \text{Id}_{E_2} \otimes \text{Id}_{E_1}$. In [4] a *geometric complexity theory* (GCT) study of Mmult and its $GL(V_1) \times GL(V_2) \times GL(V_3)$ orbit closure is considered. One sets $\mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}_3 = n$ and studies the geometry as $n \rightarrow \infty$. It is a toy case of the varieties introduced by Mulmuley and Sohoni [12, 13, 3], letting $S^d \mathbb{C}^k$ denote the homogeneous polynomials of degree d on $(\mathbb{C}^k)^*$, the varieties are $\overline{GL_{n^2} \cdot \det_n} \subset S^n \mathbb{C}^{n^2}$ and $\overline{GL_{n^2} \cdot \ell^{n-m} \text{perm}_m} \subset S^n \mathbb{C}^{n^2}$. Here $\det_n \in S^n \mathbb{C}^{n^2}$ is the determinant, a homogeneous polynomial of degree n in n^2 variables, $n > m$, $\ell \in S^1 \mathbb{C}^1$, $\text{perm}_m \in S^m \mathbb{C}^{m^2}$ is the permanent and an inclusion $\mathbb{C}^{m^2+1} \subset \mathbb{C}^{n^2}$ has been chosen. In [11] it was shown that $\text{End}_{\mathbb{C}^{n^2}} \cdot \det_n \neq \overline{GL_{n^2} \cdot \det_n}$, and determining the difference between these sets is a subject of current research.

The critical loop case with $\mathbf{e}_s = 3$ for all s is also related to the GCT program, as it corresponds to the multiplication of n matrices of size three. As a tensor, it may be thought of as a map $(X_1, \dots, X_n) \mapsto \text{trace}(X_1 \cdots X_n)$. This sequence of functions, indexed by n , considered as a sequence of homogeneous polynomials of degree n on $V_1 \oplus \cdots \oplus V_n$, is complete for the class \mathbf{VP}_e of sequences of polynomials of small formula size, see [2].

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2. CRITICAL LOOPS

Proposition 2.0.1. *Let $\mathbf{v}_1 = \mathbf{e}_2 \mathbf{e}_3$, $\mathbf{v}_2 = \mathbf{e}_3 \mathbf{e}_1$, $\mathbf{v}_3 = \mathbf{e}_2 \mathbf{e}_1$. Then $TNS(\Delta, (\mathbf{e}_2 \mathbf{e}_3, \mathbf{e}_3 \mathbf{e}_1, \mathbf{e}_2 \mathbf{e}_1), V_1 \otimes V_2 \otimes V_3)$ consists of matrix multiplication and its degenerations (and their different expressions after changes of bases), i.e.,*

$$TNS(\Delta, (\mathbf{e}_2 \mathbf{e}_3, \mathbf{e}_3 \mathbf{e}_1, \mathbf{e}_2 \mathbf{e}_1), V_1 \otimes V_2 \otimes V_3) = \text{End}(V_1) \times \text{End}(V_2) \times \text{End}(V_3) \cdot M_{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1}.$$

It has dimension $\mathbf{e}_2^2 \mathbf{e}_3^2 + \mathbf{e}_2^2 \mathbf{e}_1^2 + \mathbf{e}_3^2 \mathbf{e}_1^2 - (\mathbf{e}_2^2 + \mathbf{e}_3^2 + \mathbf{e}_1^2 - 1)$.

More generally, if Γ is a critical loop, $TNS(\Gamma, (\mathbf{e}_n \mathbf{e}_1, \mathbf{e}_1 \mathbf{e}_2, \dots, \mathbf{e}_{n-1} \mathbf{e}_n), V_1 \otimes \cdots \otimes V_n)$ is $\text{End}(V_1) \times \cdots \times \text{End}(V_n) \cdot M_{\bar{\mathbf{e}}}$, where $M_{\bar{\mathbf{e}}} : V_1 \times \cdots \times V_n \rightarrow \mathbb{C}$ is the matrix multiplication operator $(X_1, \dots, X_n) \mapsto \text{trace}(X_1 \cdots X_n)$.

Proof. For the triangle case, a generic element $T_1 \in E_2 \otimes E_3^* \otimes V_1$ may be thought of as a linear isomorphism $E_2^* \otimes E_3 \rightarrow V_1$, identifying V_1 as a space of $\mathbf{e}_2 \times \mathbf{e}_3$ -matrices, and similarly for V_2, V_3 . Choosing bases $e_s^{u_s}$ for E_s^* , with dual basis $e_{u_s, s}$ for E_s , induces bases $x_{u_3}^{u_2}$ for V_1 etc.. Let $1 \leq i \leq \mathbf{e}_2$, $1 \leq \alpha \leq \mathbf{e}_3$, $1 \leq u \leq \mathbf{e}_1$. Then

$$\text{con}(T_1 \otimes T_2 \otimes T_3) = \sum x_{\alpha}^i \otimes y_u^{\alpha} \otimes z_i^u$$

which is the matrix multiplication operator. The general case is similar. \square

Proposition 2.0.2. *The Lie algebra of the stabilizer of $M_{\mathbf{e}_n \mathbf{e}_1, \mathbf{e}_1 \mathbf{e}_2, \dots, \mathbf{e}_{n-1} \mathbf{e}_n}$ in $GL(V_1) \times \dots \times GL(V_n)$ is the image of $\mathfrak{sl}(E_1) \oplus \dots \oplus \mathfrak{sl}(E_n)$ under the map*

$$\begin{aligned} \alpha_1 \oplus \dots \oplus \alpha_n \mapsto & (Id_{E_n} \otimes \alpha_1, -\alpha_1^T \otimes Id_{E_2}, 0, \dots, 0) + (0, Id_{E_1} \otimes \alpha_2, -\alpha_2^T \otimes Id_{E_3}, 0, \dots, 0) \\ & + \dots + (-\alpha_n^T \otimes Id_{E_1}, 0, \dots, 0, Id_{E_{n-1}} \otimes \alpha_n). \end{aligned}$$

Here $\mathfrak{sl}(E_j) \subset \mathfrak{gl}(E_j)$ denotes the traceless endomorphisms and T as a superscript denotes transpose (which is really just cosmetic).

The proof is safely left to the reader.

Large loops are referred to as “1-D systems with periodic boundary conditions” in the physics literature and are often used in simulations. By Proposition 2.0.2, for a critical loop, $\dim(TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})) = \mathbf{e}_1^2 \mathbf{e}_2^2 + \dots + \mathbf{e}_{n-1}^2 \mathbf{e}_n^2 + \mathbf{e}_n^2 \mathbf{e}_1^2 - (\mathbf{e}_1^2 + \dots + \mathbf{e}_n^2 - 1)$, compared with the ambient space which has dimension $\mathbf{e}_1^2 \dots \mathbf{e}_n^2$. For example, when $\mathbf{e}_j = 2$ for all j , $\dim(TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})) = 12n + 1$, compared with $\dim \mathbf{V} = 4^n$.

3. ZARISKI CLOSURE

Theorem 3.0.3. *Let $\mathbf{v}_1 = \mathbf{e}_2 \mathbf{e}_3$, $\mathbf{v}_2 = \mathbf{e}_3 \mathbf{e}_1$, $\mathbf{v}_3 = \mathbf{e}_2 \mathbf{e}_1$. Then $TNS(\Delta, (\mathbf{e}_2 \mathbf{e}_3, \mathbf{e}_3 \mathbf{e}_1, \mathbf{e}_2 \mathbf{e}_1), V_1 \otimes V_2 \otimes V_3)$ is not Zariski closed. More generally any $TNS(\Gamma, \mathbf{e}, \mathbf{V})$ where Γ contains a cycle with no sub-critical vertex is not Zariski closed.*

Proof. Were $T(\Delta) := TNS(\Delta, (\mathbf{e}_2 \mathbf{e}_3, \mathbf{e}_3 \mathbf{e}_1, \mathbf{e}_2 \mathbf{e}_1), V_1 \otimes V_2 \otimes V_3)$ Zariski closed, it would be

$$(2) \quad \overline{GL(V_1) \times GL(V_2) \times GL(V_3) \cdot M_{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1}}.$$

To see this, note that the $G = GL(V_1) \times GL(V_2) \times GL(V_3)$ orbit of matrix multiplication is a Zariski open subset of $T(\Delta)$ of the same dimension as $T(\Delta)$.

We need to find a curve $g(t) = (g_1(t), g_2(t), g_3(t))$ such that $g_j(t) \in GL(V_j)$ for all $t \neq 0$ and $\lim_{t \rightarrow 0} g(t) \cdot M_{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1}$ is both defined and not in $End(V_1) \times End(V_2) \times End(V_3) \cdot M_{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1}$.

Note that for $(X, Y, Z) \in GL(V_1) \times GL(V_2) \times GL(V_3)$, we have $(X, Y, Z) \cdot M_{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1}(P, Q, R) = \text{trace}(X(P)Y(Q)Z(R))$. Here $X : E_2^* \otimes E_3 \rightarrow E_2^* \otimes E_3$, $Y : E_3^* \otimes E_1 \rightarrow E_3^* \otimes E_1$, $Z : E_1^* \otimes E_2 \rightarrow E_1^* \otimes E_2$.

Take subspaces $U_{E_2 E_3} \subset E_2^* \otimes E_3$, $U_{E_3 E_1} \subset E_3^* \otimes E_1$. Let $U_{E_1 E_2} := \text{Con}(U_{E_2 E_3}, U_{E_3 E_1}) \subset E_2^* \otimes E_1$ be the images of all the $pq \in E_2^* \otimes E_1$ where $p \in U_{E_2 E_3}$ and $q \in U_{E_3 E_1}$ (i.e., the matrix multiplication of all pairs of elements). Take X_0, Y_0, Z_0 respectively to be the projections to $U_{E_2 E_3}$, $U_{E_3 E_1}$ and $U_{E_1 E_2}^\perp$. Let X_1, Y_1, Z_1 be the projections to complementary spaces (so, e.g., $X_0 + X_1 = Id_{V_1^*}$). For $P \in V_1^*$, write $P_0 = X_0(P)$ and $P_1 = X_1(P)$, and similarly for Q, R .

Take the curve (X_t, Y_t, Z_t) with $X_t = \frac{1}{\sqrt{t}}(X_0 + tX_1)$, $Y_t = \frac{1}{\sqrt{t}}(Y_0 + tY_1)$, $Z_t = \frac{1}{\sqrt{t}}(Z_0 + tZ_1)$. Then the limiting tensor, as a map $V_1^* \times V_2^* \times V_3^* \rightarrow \mathbb{C}$, is

$$(P, Q, R) \mapsto \text{trace}(P_0 Q_0 R_1) + \text{trace}(P_0 Q_1 R_0) + \text{trace}(P_1 Q_0 R_0).$$

Call this tensor \tilde{M} . First observe that \tilde{M} uses all the variables (i.e., considered as a linear map $\tilde{M} : V_1^* \rightarrow V_2^* \otimes V_3$, it is injective, and similarly for its cyclic permutations). Thus it is either in the orbit of matrix multiplication or a point in the boundary that is not in $End(V_1) \times End(V_2) \times End(V_3) \cdot M_{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1}$, because all such boundary points have at least one such linear map non-injective.

It remains to show that there exist \tilde{M} such that $\tilde{M} \notin G \cdot M_{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1}$. To prove some \tilde{M} is a point in the boundary, we compute the Lie algebra of its stabilizer and show it has dimension greater than the the dimension of the stabilizer of matrix multiplication. One may take block

matrices, e.g.,

$$X_0 = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix},$$

and Y_0, Y_1 have similar shape, but Z_0, Z_1 have the shapes reversed. Here one takes any splitting $\mathbf{e}_j = \mathbf{e}'_j + \mathbf{e}''_j$ to obtain the blocks.

For another example, if one takes $\mathbf{e}_j = \mathbf{e}$ for all j , X_0, Y_0, Z_1 to be the diagonal matrices and X_1, Y_1, Z_0 to be the matrices with zero on the diagonal, then one obtains a stabilizer of dimension $4\mathbf{e}^2 - 2\mathbf{e} > 3\mathbf{e}^2 - 1$. (This example coincides with the previous one when all $\mathbf{e}_j = 2$.)

To calculate the stabilizer of \tilde{M} , first write down the tensor expression of $\tilde{M} \in V_1 \otimes V_2 \otimes V_3$ with respect to fixed bases of V_1, V_2, V_3 . Then set an equation $(X, Y, Z) \cdot \tilde{M} = 0$ where $X \in \mathfrak{gl}(V_1)$, $Y \in \mathfrak{gl}(V_2)$ and $Z \in \mathfrak{gl}(V_3)$ are unknowns. Recall that here the action of (X, Y, Z) on \tilde{M} is the Lie algebra action, so we obtain a collection of linear equations. Finally we solve this collection of linear equations and count the dimension of the solution space. This dimension is the dimension of the stabilizer of \tilde{M} in $GL(V_1) \times GL(V_2) \times GL(V_3)$.

To give an explicit example, let $\mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}_3 = \mathbf{e}$ and let $X_0 = \text{diag}(x_1^1, \dots, x_{\mathbf{e}}^{\mathbf{e}})$, $Y_0 = \text{diag}(y_1^1, \dots, y_{\mathbf{e}}^{\mathbf{e}})$, $Z_0 = \text{diag}(z_1^1, \dots, z_{\mathbf{e}}^{\mathbf{e}})$, $X_1 = (x_j^i) - X_0$, $Y_1 = (y_j^i) - Y_0$, $Z_1 = (z_j^i) - Z_0$. Then

$$\tilde{M} = \sum_{i,j=1}^{\mathbf{e}} (x_j^i y_j^j + x_i^i y_j^i) z_i^j.$$

Let $X = \sum a_{(k)}^{(j)} X_{(i)}^{(k)}$ be an element of $\mathfrak{gl}(V_1)$, where $\{X_{(i)}^{(k)}\}$ is a basis of $\mathfrak{gl}(V_1)$, and define Y and Z in the same pattern with coefficients $b_{(k)}^{(j)}$'s and $c_{(k)}^{(j)}$'s, respectively. Consider the equation $(X, Y, Z) \cdot T = 0$ and we want to solve this equation for $a_{(k)}^{(j)}$'s, $b_{(k)}^{(j)}$'s and $c_{(k)}^{(j)}$'s. For these equations to hold, the coefficients of z_i^j 's must be zero. That is, for each pair (j, i) of indices we have:

$$\sum_{k,l=1}^{\mathbf{e}} a_{(k)}^{(j)} x_l^k y_j^j + b_{(k)}^{(j)} x_j^i y_k^l + a_{(k)}^{(i)} x_l^k y_j^i + b_{(k)}^{(i)} x_i^i y_l^k + c_{(j)}^{(l)} (x_l^k y_l^l + x_k^k y_l^k) = 0.$$

For these equations to hold, the coefficients of y_s^r 's must be zero. For example, if $s \neq j$, $r \neq s$ then we have:

$$b_{(s)}^{(j)} x_j^i + b_{(s)}^{(i)} x_i^i + c_{(j)}^{(s)} x_r^r = 0$$

Now coefficients of x terms must be zero, for instance, if $i \neq j$ and $i \neq r$, then we have:

$$b_{(r)}^{(j)} = 0, \quad b_{(r)}^{(i)} = 0, \quad c_{(j)}^{(s)} = 0.$$

If one writes down and solves all such linear equations, the dimension of the solution is $4\mathbf{e}^2 - 2\mathbf{e}$.

The same construction works for larger loops and cycles in larger graphs as it is essentially local - one just takes all other curves the constant curve equal to the identity. \square

Remark 3.0.4. When $\mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}_3 = 2$ we obtain a codimension one component of the boundary. In general, the dimension of the stabilizer is much larger than the dimension of G , so the orbit closures of these points do not give rise to codimension one components of the boundary. It remains an interesting problem to find the codimension one components of the boundary.

4. ALGEBRAIC GEOMETRY PERSPECTIVE

For readers familiar with algebraic geometry, we recast the previous section in the language of algebraic geometry and put it in a larger context. This section also serves to motivate the proof of the previous section.

To make the parallel with the GCT program clearer, we describe the Zariski closure as the cone over the (closure of) the image of the rational map (i.e., the “closure” of the map defined on a Zariski open subset)

$$(3) \quad \mathbb{P}End(V_1) \times \mathbb{P}End(V_2) \times \mathbb{P}End(V_3) \dashrightarrow \mathbb{P}(V_1 \otimes V_2 \otimes V_3) \\ ([X], [Y], [Z]) \mapsto (X, Y, Z) \cdot [M_{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1}].$$

(Compare with the map ψ in [3, §7.2].) A dashed arrow is used to indicate the map is not everywhere defined.

The indeterminacy locus (that is, points $([X], [Y], [Z])$ where the map is not defined), consists of $([X], [Y], [Z])$ such that for all triples of matrices P, Q, R , $\text{trace}(X(P)Y(Q)Z(R)) = 0$. In principle one can obtain (2) as the image of a map from a succession of blow-ups of $\mathbb{P}End(V_1) \times \mathbb{P}End(V_2) \times \mathbb{P}End(V_3)$. (See, e.g., [8, p. 81] for the definition of a blow-up)

One way to attain a point in the indeterminacy locus is to take $([X_0], [Y_0], [Z_0])$ as described in the proof. Taking a curve in G that limits to this point may or may not give something new. In the proof we gave two explicit choices that do give something new.

A more invariant way to discuss that $\tilde{M} \notin End(V_1) \times End(V_2) \times End(V_3) \cdot M_{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1}$ is to consider an auxiliary variety, called a *subspace variety*,

$$Sub_{\mathbf{f}_1, \dots, \mathbf{f}_n}(\mathbf{V}) := \{T \in V_1 \otimes \dots \otimes V_n \mid \exists V'_j \subset V_j, \dim V'_j = \mathbf{f}_j, \text{ and } T \in V'_1 \otimes \dots \otimes V'_n\},$$

and observe that if $T \in \times_j End(V_j) \cdot M_{\tilde{\mathbf{e}}}$ and $T \notin \times_j GL(V_j) \cdot M_{\tilde{\mathbf{e}}}$, then $T \in Sub_{\mathbf{f}_1, \dots, \mathbf{f}_n}(\mathbf{V})$ where $\mathbf{f}_j < \mathbf{e}_j$ for at least one j .

The statement that “ \tilde{M} uses all the variables” may be rephrased as saying that $\tilde{M} \notin Sub_{\mathbf{e}_2\mathbf{e}_3-1, \mathbf{e}_2\mathbf{e}_1-1, \mathbf{e}_3\mathbf{e}_1-1}(V_1 \otimes V_2 \otimes V_3)$

5. REDUCTION FROM THE SUPERCRITICAL CASE TO THE CRITICAL CASE WITH THE SAME GRAPH

For a vector space W , let $G(k, W)$ denote the Grassmannian of k -planes through the origin in W . Let $\mathcal{S} \rightarrow G(k, W)$ denote the tautological rank k vector bundle whose fiber over $E \in G(k, W)$ is the k -plane E . Assume $\mathbf{f}_j \leq \mathbf{v}_j$ for all j with at least one inequality strict. Form the vector bundle $\mathcal{S}_1 \otimes \dots \otimes \mathcal{S}_n$ over $G(\mathbf{f}_1, V_1) \times \dots \times G(\mathbf{f}_n, V_n)$, where $\mathcal{S}_j \rightarrow G(\mathbf{f}_j, V_j)$ are the tautological subspace bundles. Note that the total space of $\mathcal{S}_1 \otimes \dots \otimes \mathcal{S}_n$ maps to \mathbf{V} with image $Sub_{\tilde{\mathbf{f}}}(\mathbf{V})$. Define a fiber sub-bundle, whose fiber over $(U_1 \times \dots \times U_n) \in G(\mathbf{f}_1, V_1) \times \dots \times G(\mathbf{f}_n, V_n)$ is $TNS(\Gamma, \tilde{\mathbf{e}}, U_1 \otimes \dots \otimes U_n)$. Denote this bundle by $TNS(\Gamma, \tilde{\mathbf{e}}, \mathcal{S}_1 \otimes \dots \otimes \mathcal{S}_n)$.

The supercritical cases may be realized, in the language of Kempf, as a “collapsing of a bundle” over the critical cases as follows:

Proposition 5.0.5. *Assume $\mathbf{f}_j := \Pi_{s \in e(j)} \mathbf{e}_s \leq \mathbf{v}_j$. Then $TNS(\Gamma, \tilde{\mathbf{e}}, \mathbf{V})$ is the image of the bundle $TNS(\Gamma, \tilde{\mathbf{e}}, \mathcal{S}_1 \otimes \dots \otimes \mathcal{S}_n)$ under the map to \mathbf{V} . In particular*

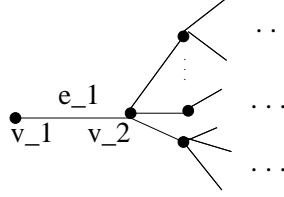
$$\dim(TNS(\Gamma, \tilde{\mathbf{e}}, \mathbf{V})) = \dim(TNS(\Gamma, \tilde{\mathbf{e}}, \mathbb{C}^{\mathbf{f}_1} \otimes \dots \otimes \mathbb{C}^{\mathbf{f}_n})) + \sum_{j=1}^n \mathbf{f}_j(\mathbf{v}_j - \mathbf{f}_j).$$

Proof. If $\Pi_{s \in e(j)} \mathbf{e}_s \leq \mathbf{v}_j$, then any tensor $T \in V_j \otimes (\otimes_{s \in in(j)} E_s) \otimes (\otimes_{t \in out(j)} E_t^*)$, must lie in some $V'_j \otimes (\otimes_{s \in in(j)} E_s) \otimes (\otimes_{t \in out(j)} E_t^*)$ with $\dim V'_j = \mathbf{f}_j$. The space $TNS(\Gamma, \tilde{\mathbf{e}}, \mathbf{V})$ is the image of this subbundle under the map to \mathbf{V} . \square

This type of bundle construction is standard, see [10, 16]. Using the techniques in [16], one may reduce questions about a supercritical case to the corresponding critical case.

6. REDUCTION OF CASES WITH SUBCRITICAL VERTICES OF VALENCE ONE

The subcritical case in general can be understood in terms of projections of critical cases, but this is not useful for extracting information. However, if a subcritical vertex has valence one, one may simply reduce to a smaller graph as we now describe.



Proposition 6.0.6. *Let $TNS(\Gamma, \vec{e}, \mathbf{V})$ be a tensor network state, let v be a vertex of Γ with valence one. Relabel the vertices such that $v = v_1$ and so that v_1 is attached by e_1 to v_2 . If $\mathbf{v}_1 \leq \mathbf{e}_1$, then $TNS(\Gamma, \vec{e}, V_1 \otimes \cdots \otimes V_n) = TNS(\tilde{\Gamma}, \vec{e}, \tilde{V}_1 \otimes V_3 \otimes \cdots \otimes V_n)$, where $\tilde{\Gamma}$ is Γ with v_1 and e_1 removed, \vec{e} is the vector $(\mathbf{e}_2, \dots, \mathbf{e}_n)$ and $\tilde{V}_1 = V_1 \otimes V_2$.*

Proof. A general element in $TNS(\Gamma, \vec{e}, V_1 \otimes \cdots \otimes V_n)$ is of the form $\sum_{i,j=1}^{\mathbf{e}_1, \mathbf{e}_2} u_i \otimes v_{iz} \otimes w_z$, where $w_z \in V_3 \otimes \cdots \otimes V_n$. Obviously, $TNS(\Gamma, \vec{e}, V_1 \otimes \cdots \otimes V_n) \subseteq TNS(\tilde{\Gamma}, \vec{e}, \tilde{V}_1 \otimes V_3 \otimes \cdots \otimes V_n) =: TNS(\tilde{\Gamma}, \vec{e}, \tilde{\mathbf{V}})$. Conversely, a general element in $TNS(\tilde{\Gamma}, \vec{e}, \tilde{\mathbf{V}})$ is of the form $\sum_z X_z \otimes w_z$, $X_z \in V_1 \otimes V_2$. Since $\mathbf{v}_1 \leq \mathbf{e}_1$, we may express X_z in the form $\sum_{i=1}^{\mathbf{e}_1} u_i \otimes v_{iz}$, where u_1, \dots, u_{v_1} is a basis of V_1 . Therefore, $TNS(\Gamma, \vec{e}, \mathbf{V}) \supseteq TNS(\tilde{\Gamma}, \vec{e}, \tilde{\mathbf{V}})$. \square

7. TREES

With trees one can apply the two reductions successively to reduce to a tower of bundles where the fiber in the last bundle is a linear space. The point is that a critical vertex is both sub- and supercritical, so one can reduce at valence one vertices iteratively. Here are a few examples in the special case of chains. The result is similar to the Allman-Rhodes reduction theorem for phylogenetic trees [1].

Example 7.0.7. Let Γ be a chain with 3 vertices. If it is supercritical, $TNS(\Gamma, \vec{e}, \mathbf{V}) = V_1 \otimes V_2 \otimes V_3$. Otherwise $TNS(\Gamma, \vec{e}, \mathbf{V}) = \text{Sub}_{\mathbf{e}_1, \mathbf{e}_1 \mathbf{e}_2, \mathbf{e}_2}(V_1 \otimes V_2 \otimes V_3)$.

Example 7.0.8. Let Γ be a chain with 4 vertices. If $\mathbf{v}_1 \leq \mathbf{e}_1$ and $\mathbf{v}_4 \leq \mathbf{e}_3$, then, writing $W = V_1 \otimes V_2$ and $U = V_3 \otimes V_4$, by Proposition 6.0.6, $TNS(\Gamma, \vec{e}, \mathbf{V})$ is the set of rank at most \mathbf{e}_2 elements in $W \otimes U$ (the secant variety of the two-factor Segre). Other chains of length four have similar complete descriptions.

Example 7.0.9. Let Γ be a chain with 5 vertices. Assume that $\mathbf{v}_1 \leq \mathbf{e}_1$, $\mathbf{v}_5 \leq \mathbf{e}_4$ and $\mathbf{v}_1 \mathbf{v}_2 \geq \mathbf{e}_2$ and $\mathbf{v}_4 \mathbf{v}_5 \geq \mathbf{e}_3$. Then $TNS(\Gamma, \vec{e}, \mathbf{V})$ is the image of a bundle over $G(\mathbf{e}_2, V_1 \otimes V_2) \times G(\mathbf{e}_3, V_4 \otimes V_5)$ whose fiber is the set of tensor network states associated to a chain of length three.

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